Numerical Linear Algebra
Where?
When?
How?

Negin Bagherpour

November 2020

Numerical Linear Algebra Group
Department of Mathematics, The University of Manchester
https://nla-group.org/

Richard Harold Bartels

MathSciNet

Ph.D. Stanford University 1968

Dissertation:

Advisor: Gene Howard Golub

Students:
Click here to see the students listed in chronological order.

Name	School	Year	Descendants
Busovaca, Sead	University of Waterloo	1985	
Forsey, David	University of Waterloo	1990	
Mahdavi-Amiri, Nezameddin	The Johns Hopkins University	1981	49
Mutrie, Mark	University of Waterloo	1992	
Vermeulen, Alan	University of Waterloo	1995	

According to our current on-line database, Richard Bartels has 5 students and 54 descendants.
We welcome any additional information.

If you have additional information or corrections regarding this mathematician, please use the update form. To submit students of this mathematician, please use the new data form, noting this mathematician's MGP ID of 46401 for the advisor ID.

Purdue University Numerical Linear Algebra Group (PUNLAG)
Department of Mathematics, Purdue University
https://www.math.purdue.edu/~kkloste/punlag.html

Machine Learning	Optimization	PDE
Dataset and Data Files	Modeling	Finite Difference Method
Images and Photographs	Optimality Conditions	Finite Element Method
One-Hot Encoding	Active Set	System Dynamics
Linear Regression	Hessian Approximation	
Regularization	SQP Method	
Principal Component Analysis	Interior point Methods	
Singular-Value Decomposition	Nonlinear Matrix Equations	
Latent Semantic Analysis	Specific Objectives (MR)	
Recommender Systems		
Deep Learning		
Page Rank		

Machine Learning

- Dataset and Data Files

Price dataset

Date	Min_Price	Max_Price	Close
2020.10.31	65473	65722	65720
2020.11.01	64869	66125	65946
2020.11.02	65835	66358	66128

Sampling, variety of characteristics

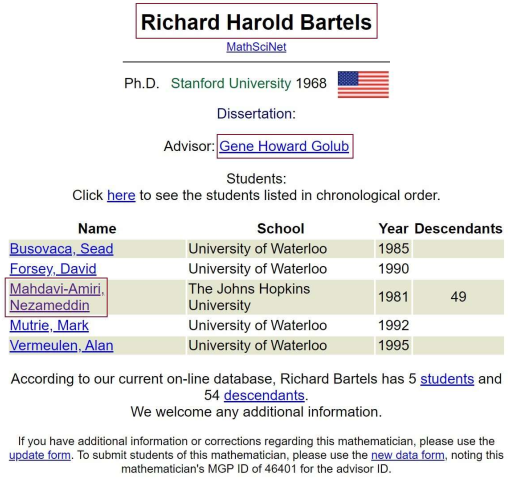
	Scan improvement	Pain	Nausea	Edema
Dosage 1	+	+	++	+
Dosage 2	+	+	+	+
Dosage 3	+	+	+	-
Dosage 4	-	+	-	-

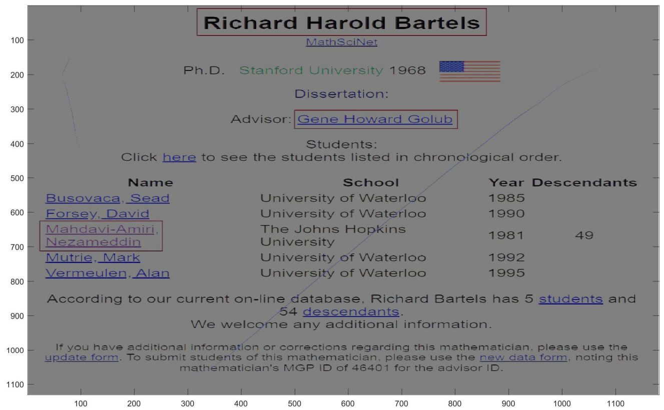
Machine Learning

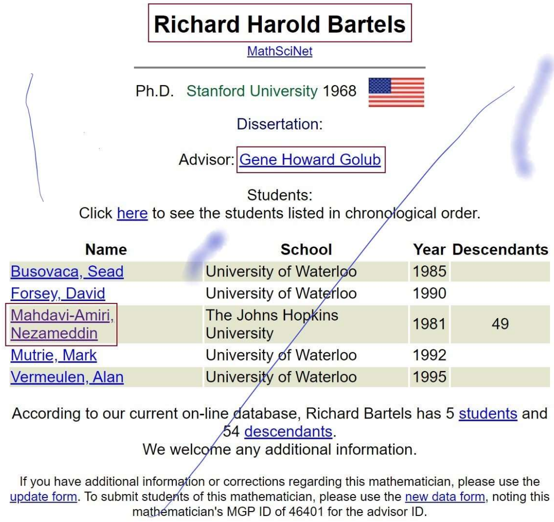
- Images and Photographs

  <u>Processing</u>

  A=imread('golub.jpg') and image(A)







Machine Learning

- One-Hot Encoding

red
green
blue

red	green	blue
1	0	0
0	1	0
0	0	1

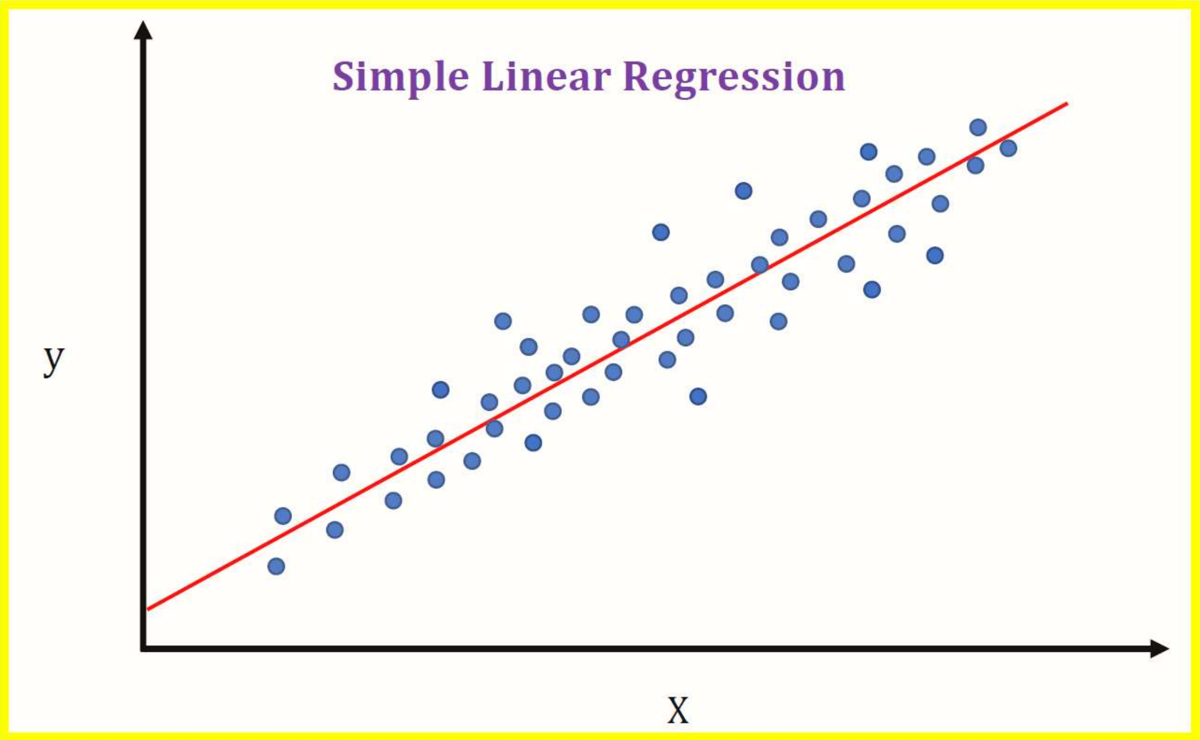
Machine Learning

- Linear Regression

Python codes

2-dimentional form

$$Y = AX$$



Machine Learning

- Principal Component Analysis

Model order reduction
Eigenvalues decomposition

Machine Learning

- Singular-Value Decomposition

Model order reduction
Feature selection
Visualization
Noise reduction

Machine Learning

- Latent Semantic Analysis

  Natural language processing
  Sparse matrix
  Columns: words
  Rows: address

Machine Learning

- Recommender Systems

  Product recommendation based on previous choices

    Selection similarity between people

Machine Learning

- Deep Learning

  Design of neural networks

Machine Learning

- Page Rank

$$PR(p_i) = \frac{1-d}{N} + d \sum_{p_j \in M(p_i)} \frac{PR(p_j)}{L(p_j)}$$

$$\mathbf{R} = \begin{bmatrix} (1-d)/N \\ (1-d)/N \\ \vdots \\ (1-d)/N \end{bmatrix} + d \begin{bmatrix} \ell(p_1,p_1) & \ell(p_1,p_2) & \cdots & \ell(p_1,p_N) \\ \ell(p_2,p_1) & \ddots & & \vdots \\ \vdots & & \ell(p_i,p_j) & \\ \ell(p_N,p_1) & \cdots & & \ell(p_N,p_N) \end{bmatrix} \mathbf{R}$$

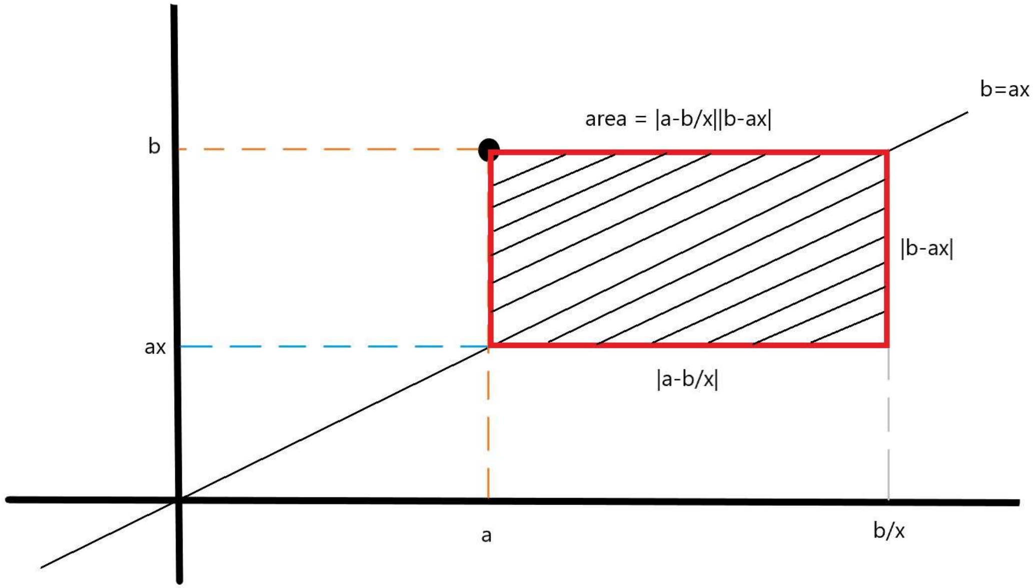
Optimization

- Modeling

$$\min tr(DX-T)^T(DX-T)$$

$$X \succ 0$$

$$XAX = B$$



$$area = \frac{(ax-b)^2}{x}, x > 0$$

Optimization

- Modeling

Inverse approximation

$$\min \|AX - I\|^2$$

Steepest descent and BB step length

Optimization

- Optimality conditions

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{z}) \\ \text{subject to} \quad & f_i(\mathbf{z}) \leq 0, \quad i = 1, \cdots, m \\ & h_i(\mathbf{z}) = 0, \quad i = 1, \cdots, p \\ & \|z - x\| \leq R \end{aligned}$$

$$u_0 \nabla f_0(\bar{x}) + \sum_{i=1}^{m} u_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} v_i \nabla h_i(\bar{x}) = 0$$

$$(u_0, u) \succeq 0, \quad (u_0, u, v) \neq 0$$

$$u_i f_i(\bar{x}) = 0, \quad i = 1, \cdots, m$$

Second order necessary condition

$$\nabla f(\bar{\mathbf{x}}) = \mathbf{0} \quad \textit{and} \quad \nabla^2 f(\bar{\mathbf{x}}) \succeq \mathbf{0}$$

KKT sufficient conditions (convex)

$$\nabla f_0(\bar{x}) + \sum_{i=1}^{m} u_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} v_i \nabla h_i(\bar{x}) = 0$$

$$u \succeq 0, \quad u_i f_i(\bar{x}) = 0, \quad i = 1, \cdots, m$$

Optimization

- Active set

$$x \in \mathbb{R}^{n+1}: \quad \begin{array}{ll} \min f(x) + \rho x_{n+1} & \\ g_j(x) = 0\ , & j = 1, \ldots, m_e\ , \\ g_j(x) + x_{n+1} \geq 0\ , & j = m_e + 1, \ldots, m\ , \\ x_{n+1} \geq 0 & \end{array}$$

$$d \in \mathbb{R}^n, \delta \in \mathbb{R}: \quad \begin{array}{l} \min\ \frac{1}{2} d^T B_k d + \nabla f(x_k)^T d\ + \frac{1}{2}\sigma_k^2 \delta^2\ , \\ \nabla g_j(x_k)^T d + (1-\delta) g_j(x_k) \left\{ \begin{matrix} = \\ \geq \end{matrix} \right\} 0\ ,\ j \in J_k^\star\ , \\ \nabla g_j(x_{j(k)})^T d + g_j(x_k) \geq 0\ ,\ j \in \overline{K}_k^\star \end{array}$$

$$J_k^\star \doteq \{1, \ldots, m_e\} \cup \{j : m_e < j \leq m,\ g_j(x_k) < \epsilon \text{ or } v_j^{(k)} > 0\}$$

Optimization

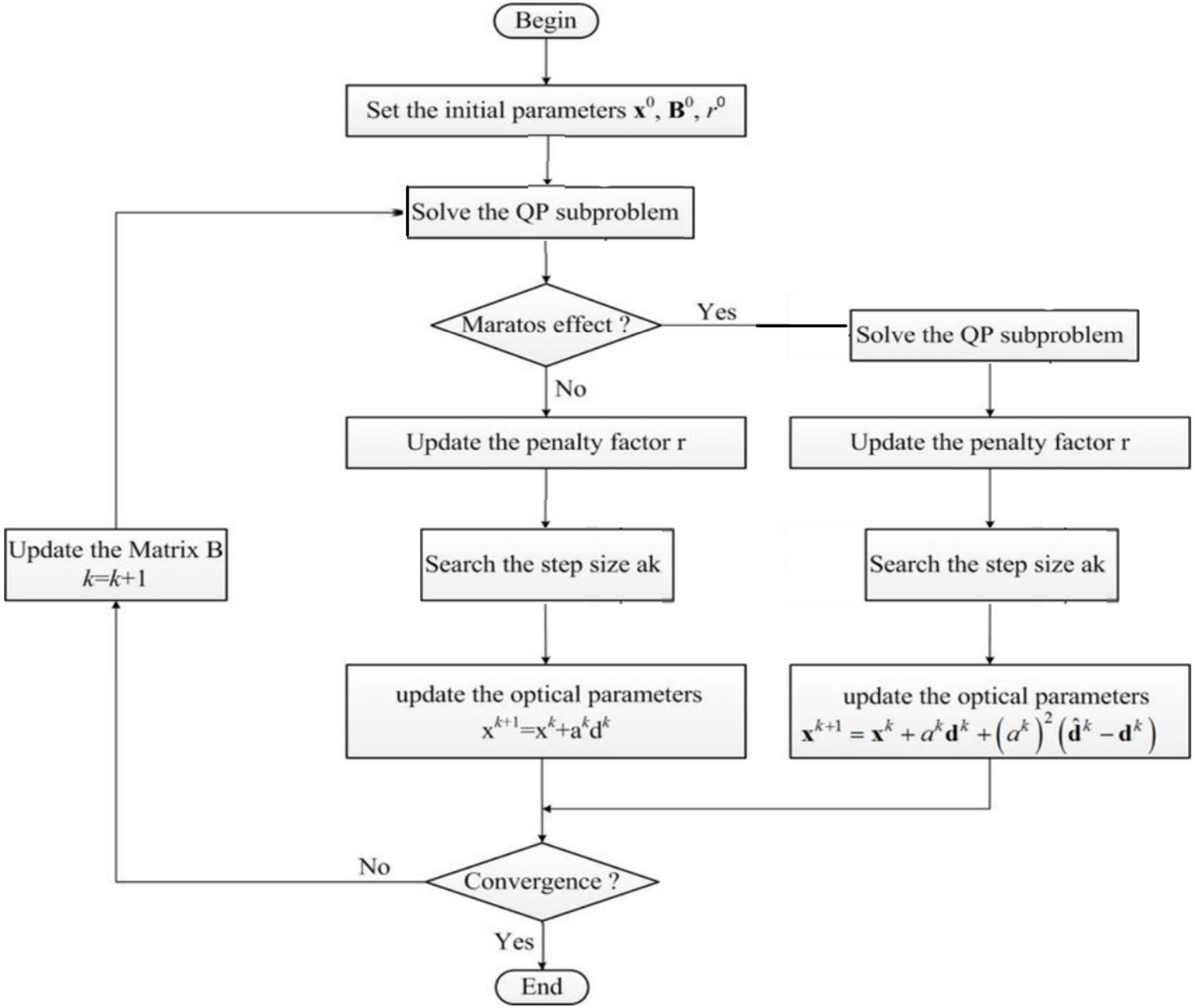
- Hessian Approximation - BFGS

$$
\begin{aligned}
x^{(k+1)} &= x^{(k)} - H_k^{-1} \nabla f(x^{(k)}), \\
s^{(k)} &= x^{(k+1)} - x^{(k)}, \\
y^{(k)} &= \nabla f(x^{(k+1)}) - \nabla f(x^{(k)}), \\
H_{k+1} &= H_k - \frac{H_k s^{(k)} (s^{(k)})^T H_k}{s^{(k)} \cdot H_k s^{(k)}} + \frac{y^{(k)} (y^{(k)})^T}{y^{(k)} \cdot s^{(k)}}
\end{aligned}
$$

$$Hs = y$$

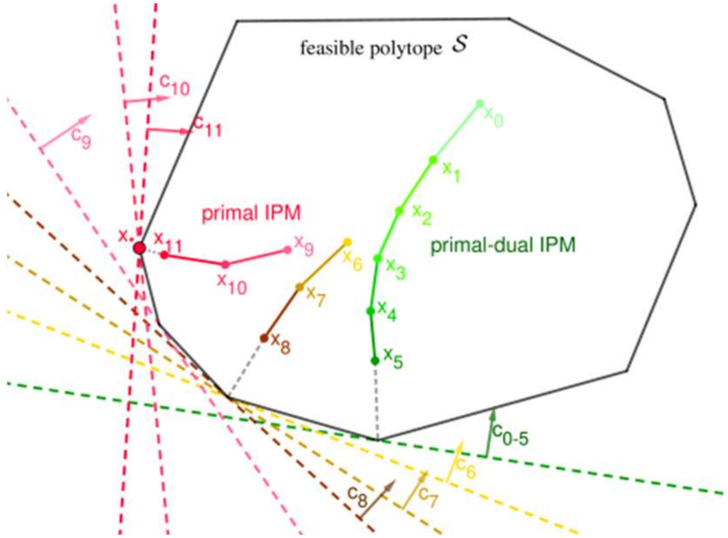
Optimization

- SQP



Optimization

- Interior Point Method



Perturbed KKT

$$r(x,u,v) = \begin{pmatrix} \nabla f(x) + Dh(x)^T u + A^T v \\ -\mathrm{diag}(u)h(x) - (1/t)1 \\ Ax - b \end{pmatrix} = 0$$

$$h(x) = \begin{pmatrix} h_1(x) \\ \dots \\ h_m(x) \end{pmatrix}, \quad Dh(x) = \begin{bmatrix} \nabla h_1(x)^T \\ \dots \\ \nabla h_m(x)^T \end{bmatrix}$$

Optimization

- Nonlinear matrix equation

$$X - \sum_{i=1}^{m} A_i^{\star} F(X) A_i = Q,$$

Linearization
Newton approach
PDEIV approach

Optimization

- Minimum rank

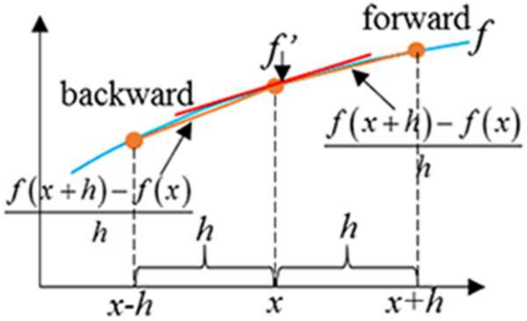
$$\min rank(X)$$

$$||AX - B|| < \delta$$

Control
Stability
Graph theory

PDE

- <u>FDM</u>



$$f_x(x,y) \approx \frac{f(x+h,y)-f(x-h,y)}{2h}$$

$$f_y(x,y) \approx \frac{f(x,y+k)-f(x,y-k)}{2k}$$

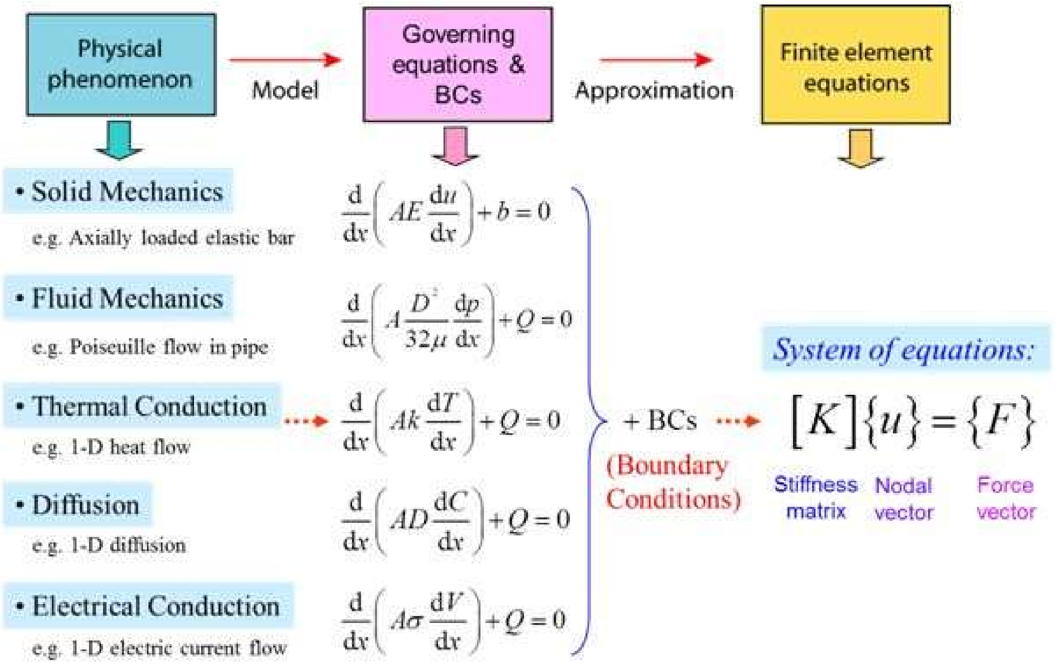
$$f_{xx}(x,y) \approx \frac{f(x+h,y)-2f(x,y)+f(x-h,y)}{h^2}$$

$$f_{yy}(x,y) \approx \frac{f(x,y+k)-2f(x,y)+f(x,y-k)}{k^2}$$

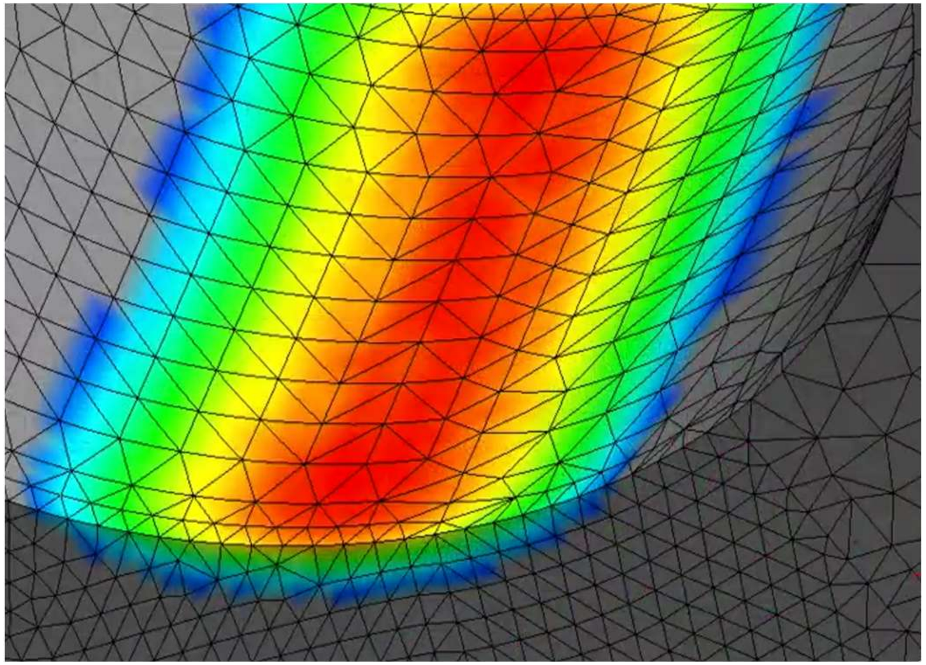
$$f_{xy}(x,y) \approx \frac{f(x+h,y+k)-f(x+h,y-k)-f(x-h,y+k)+f(x-h,y-k)}{4hk}.$$

PDE

- FEM

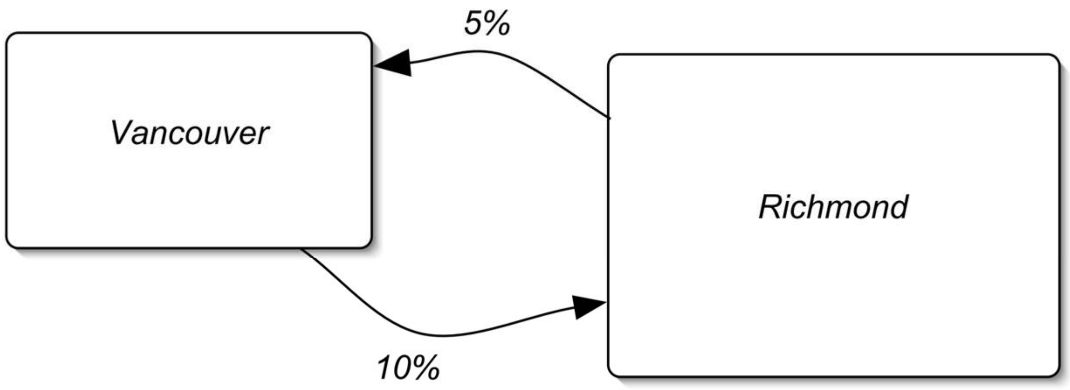


Physical phenomenon
Model
Governing equations & BCs
Approximation
Finite element equations
• Solid Mechanics
e.g. Axially loaded elastic bar
• Fluid Mechanics
e.g. Poiseuille flow in pipe
• Thermal Conduction
e.g. 1-D heat flow
• Diffusion
e.g. 1-D diffusion
• Electrical Conduction
e.g. 1-D electric current flow
$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b=0$
$\frac{d}{dx}\left(A\frac{D^2}{32\mu}\frac{dp}{dx}\right)+Q=0$
$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+Q=0$
$\frac{d}{dx}\left(AD\frac{dC}{dx}\right)+Q=0$
$\frac{d}{dx}\left(A\sigma\frac{dV}{dx}\right)+Q=0$
+ BCs
(Boundary Conditions)
System of equations:
$[K]\{u\}=\{F\}$
Stiffness matrix
Nodal vector
Force vector



PDE

- System dynamics



$$v_{2008} = .9\, v_{2007} + .05\, r_{2007}$$
$$r_{2008} = .1\, v_{2007} + .95\, r_{2007}$$
$$v_{n+1} = .9\, v_n + .05\, r_n$$
$$r_{n+1} = .1\, v_n + .95\, r_n$$

$$\begin{pmatrix} v_{n+1} \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} .9\, v_n + .05\, r_n \\ .1\, v_n + .95\, r_n \end{pmatrix} = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v_n \\ r_n \end{pmatrix}$$

$$\dot{X} = AX$$

Thank U