Game Theory - Week 2

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Overview

Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- **Idea**: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i .
  - Pure Strategy: only one action is played with positive probability
  - Mixed Strategy: more than one action is played with positive probability
    - These actions are called the support of the mixed strategy
- Let the set of all strategies for i be S_i
- Let the set of all strategies profiles be $S = S_1 \times \cdots \times S_n$

Utility Under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
    - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Best Response and Nash Equilibrium

- Our definition of best response and Nash equilibrium generalize from action to strategies.

Definition (Best Response)

$s_i^* \in BS(s_{-i})$ if $\forall\, s_i \in S_i,\ \ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.

Definition (Nash Equilibrium)

$s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium if $\forall i,\ \ s_i \in BR(s_{-i})$.

Theorem (Nash, 1950)

Every finite game has a Nash equilibrium.

Equality of Payoff Theorem

Equality of Payoff Theorem

Suppose $\hat{\sigma}_R$ is the best response to σ_C and i, j are strategies, then we have:

- $\hat{\sigma}_R(i), \hat{\sigma}_R(j) > 0 \implies \mathrm{P}_R\sigma_C(i) = P_R\sigma_C(j)$
- $\hat{\sigma}_R(i) > 0, \hat{\sigma}_R(j) = 0 \implies \mathrm{P}_R\sigma_C(i) \geq P_R\sigma_C(j)$

Example - Soccer Penalty Kicks

In soccer penalty kicks, a kicker has to try kick the ball into the goal and the goalie try to move to deflect the ball. this happens very quickly, so it's essentially a simultaneous move game. in our simplified version, the kicker is choosing kick either to the right or left and goalie decide to dive either to the right or left to deflect the ball.

Kicker/Goalie	Left	Right
Left	$0, 1$	$1, 0$
Right	$0.75, 0.25$	$0, 1$

Example - Soccer Penalty Kicks (Continued)

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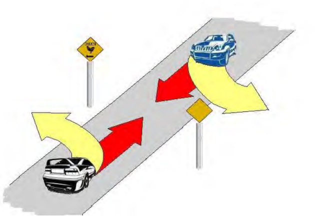
- Goalie's strategy adjusts, and the kicker actually adjusts to kick more to their weak side!
- Similarly, q probability kicker kicks left; goalie indifferent:

$$1q + 0.25(1-q) = 0q + 1(1-q) \implies q = 3/5$$

Example - Chicken

Each driver may chicken out and swerve or can decide to keep driving. If one chickens out and swerves, but the other does not, then it is a great success for the player with iron nerves and a great disgrace for the chicken. If both players have iron nerves, disaster strikes (and both incur a large penalty M)



	S (Swerve)	D (Drive)
S	(1,1)	$(-1,2)$
D	$(2,-1)$	$(-M,-M)$

Example - Chicken (Continued)

This game is a symmetric game. In general, we have:

Definition (Symmetric Game and Symmetric Equilibrium)

A normal-form game is symmetric if and only if the players have identical strategy spaces ($S_1 = S_2 = \ldots = S_n = S$) and $u_i(s_i, s_{-i})$ $= u_j(s_j, s_{-j})$, for $s_i = s_j$ and $s_{-i} = s_{-j}$ for all $i, j \in \{1, \ldots, n\}$. Also, we refer to a strategy profile with all players playing the same strategy as a symmetric profile, or, if such a profile is a Nash equilibrium, a symmetric equilibrium.

Theorem

A finite symmetric game has a symmetric mixed-strategy equilibrium.

Example - Chicken (Continued)

So what are the nash equilibria here?

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- There are two pure Nash equilibria in this game, (S, D) and (D, S).
- To determine the mixed equilibria, suppose that player I plays S with probability x and D with probability $1 - x$. We seek an equilibrium where player II has a positive probability on each of S and D. Thus we use equalizing payoffs theorem and have:

$$x+(1-x)\times(-1) = 2x+(1-x)\times(-M) \implies 2x-1 = (M+2)x-M$$

$$\implies x = 1 - \frac{1}{M}$$

Example - Chicken (Continued)

- Using the payoff equalization for the other player, we can see that this mixed-strategy equilibrium is a symmetric mixed-strategy equilibrium which confirms the already stated theorem.

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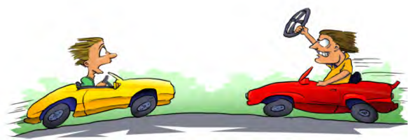
- Even though both payoff matrices decrease as M increases, the equilibrium payoffs increase. This contrasts with zero-sum games where decreasing a player's payoff matrix can only lower her expected payoff in equilibrium.

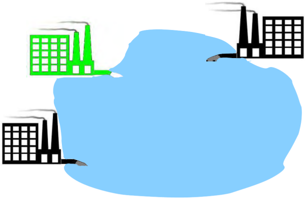
Example - Chicken (Continued)

- The payoff for a player is lower in the symmetric Nash equilibrium than it is in pure equilibrium. One way for a player to ensure3 that the higher payoff asymmetric Nash equilibrium is reached is to irrevocably commit to strategy D, for example, by ripping out the steering wheel and throwing it out of the car. In a number of games, making this kind of binding commitment pushes the game into a pure Nash equilibrium, and the nature of that equilibrium strongly depends on who managed to commit first.



Example - Pollution Game

Three firms will either pollute a lake in the following year or purify it. They pay a cost to purify, but it is free to pollute. If two or more pollute the lake, then the water in the lake is useless, and each firm must pay a cost to obtain the water that they need from elsewhere. If at most one firm pollutes, then the water is usable, and the firms incur no further costs.



Example - Pollution Game (Continued)

- If firm III purifies, the cost matrix (cost = -payoff) is as follows:

Firm II		purify	pollute
	purify	(1,1,1)	(1,0,1)
	pollute	(0,1,1)	(3,3,4)

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- If firm III pollutes, then it is:

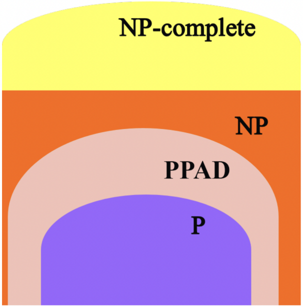
		purify	pollute
Firm II	purify	(1,1,0)	(4,3,3)
	pollute	(3,4,3)	(3,3,3)

This is a game with more than 2 players. what are the Nash equilibria here?

Complexity Class of Nash Equilibrium

So, how hard is it to find a Nash equilibrium?

- Finding Nash equilibrium is hard, but it can be done in PPAD (Polynomial Parity Arguments on Directed graphs) time.



- However, it is NP-Complete to find a “tiny” bit more info than a Nash equilibrium. For example, the following are NP-Complete:

  1. **(Uniqueness)** Given a game G, does there exist a unique equilibrium in G?
  2. **(Pareto Optimality)** Given a game G, does there exist a strictly Pareto efficient equilibrium in G?
  3. **(Guaranteed Payoff)** Given a game G and a value v, does there exist an equilibrium in G in which some player i obtains an expected payoff of at least v?

4. **(Guaranteed Social Welfare)** Given a game G, does there exist an equilibrium in which the sum of agents' utilities is at least k?
5. **(Action Inclusion)** Given a game G and an action $a_i \in A_i$ for some player $i \in \mathbb{N}$, does there exist an equilibrium of G in which player i plays action a_i with strictly positive probability?
6. **(Action Exclusion)** Given a game G and an action $a_i \in A_i$ for some player $i \in \mathbb{N}$, does there exist an equilibrium of G in which player i plays action a_i with zero probability?